

Recovering Infinities in Graviton Scattering Amplitudes using Cutkosky rules.

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Abstract

We use the Cutkosky rules as a tool for determining the infinities present in graviton scattering amplitudes. We are able to confirm theoretical derivations of counterterms in Einstein-Maxwell theory and to determine new results in the Dirac-Einstein counter-Lagrangian.

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One of the central issues in perturbative quantum gravity [1] and supergravity [2] is that of renormalisation. The dimensionful coupling constant in such theories implies that new infinities could be introduced at each perturbative order. Each new infinity would require new terms to be added to the counter-Lagrangian. Clearly, the explicit evaluation of ultra-violet infinities is a necessary step in determining the structure of these theories.

In some cases simple arguments about the symmetries of a theory can show that all counterterms vanish (for example, for on-shell pure gravity at one loop). However, in general, the counter-Lagrangian can only be found by calculation. Unfortunately, the well-known difficulties involved in perturbative gravity calculations impede such investigations significantly. Calculations *have* been done at one-loop for gravity coupled to matter and at two-loops for pure gravity. Where that matter is purely bosonic the full counter-Lagrangian can be determined using an algorithm due to 't Hooft and Veltman [3]. This method examines counterterms in the effective action using the background field method [4]. It has been used successfully to examine gravity coupled to scalars [3] and spin-one particles [5,6]. In all cases the theories were shown to be non-renormalisable. The 't Hooft-Veltman algorithm is not so useful in theories involving fermions. However, in the Dirac-Einstein system some progress has been made via direct calculation [7]; by focussing on the infinities in a well-chosen amplitude it was possible to determine one of the counterterms. While this is sufficient to show that the theory is non-renormalisable, no expression for the complete counter-Lagrangian has been constructed. In supergravity theories three-loop results are required to decide renormalisability; due to the difficulty of the calculations involved, such results remain undetermined. In this paper we will determine ultra-violet infinities directly from physical amplitudes using the Cutkosky rules.

The Cutkosky rules [8,9] are a useful way to obtain information about amplitudes; they allow us to construct expressions containing the correct cuts in all channels. In a recent paper [10] it was shown how these rules can provide information about the one-loop divergences: Since logarithmic terms can be found exactly from the Cutkosky-based calculations, and these only appear as part of the expansion

$$(-s_{ij})^{-\epsilon} = 1 - \epsilon \ln(-s_{ij}) + \frac{1}{2}\epsilon^2 \ln(-s_{ij})^2 + \dots \quad (1)$$

at one-loop, it is a straightforward task to deduce all infinite contributions. (We work in the 'four dimensional helicity' form of dimensional regularisation [11] with $\epsilon = 2 - D/2$.) Since the IR divergences can be derived independently, this method can be used to identify one-loop UV divergences in a relatively efficient way. This was used to confirm previous derivations of counterterms for gravity coupled to scalars. In this paper we will apply the same method to cases with gravity coupled to fermions and photons. In the former case this will enable us to find new information about the Dirac-Einstein counter-Lagrangian. In the latter we will show how the supersymmetric Ward identities [12,13,14] can be used

to further simplify this method; our result will confirm the previous derivation of the Einstein-Maxwell counterterms.

Gravity Coupled to Fermions

Let us begin with the theory with gravity coupled to fermions [7]. The process for obtaining cuts in this case follows the scalar calculation [10] very closely. The simplest case we can consider is the amplitude with four external fermions. Rather than calculate this for a general case, let us look at specific choices of external particle helicity. There are three independent helicity configurations which can be considered: $(+, +, +, +)$, $(-, +, +, +)$ and $(-, +, -, +)$. It is easy to show that all cuts vanish for the first two, implying that these amplitudes will be divergence free, so let us concentrate on the case $(1^-, 2^+, 3^-, 4^+)$ (with chiral fermions).

For a full cut calculation we must sum over all internal states. In this cases that means including both graviton and fermion contributions (fig. A) and considering all internal helicity configurations. Since there is an asymmetry amongst the external helicities which must also look for different contributions in different channels. (In fact we need only consider the u - and s -channels; the ‘ s ’ and ‘ t ’ calculations *will* be related by symmetry.)

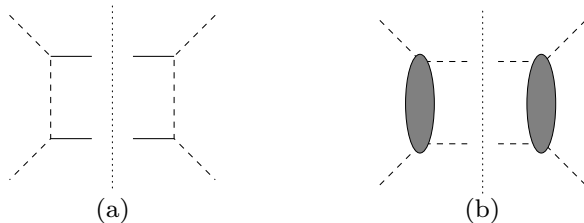


Figure A: Cut contributions required for the four fermion calculation.

First, consider the u -channel. In this case the internal graviton contribution is trivial; there is no internal helicity configuration for which all the trees involved in the calculation are non-zero. So, we only need to calculate cuts with internal fermions. The building block for this case will be the tree with four external fermions. The only choice of external helicities for which the tree is non-zero is $A^{\text{tree}}(-, +, -, +)$. We can find this from direct calculation:

$$A(1^-, 2^+, 3^-, 4^+) = -\frac{i \langle 13 \rangle [24] \kappa^2}{8 st} (2s^2 + st + 2t^2). \quad (2)$$

(Note that throughout this work we use the spinor-helicity notation of refs. [15,13].) Sewing two such trees together, expanding this and integrating gives us the u -channel cut contri-

bution to be:

$$-\frac{i r_{\Gamma} \kappa^4}{(4\pi)^{2-\epsilon}} \frac{1}{16 \langle 24 \rangle [13]} \left(\frac{u^3 (2t^2 + st + 2s^2) \ln(u)}{st} \frac{1}{\epsilon} + \frac{2 u^4 \ln(u) \ln(t)}{t} + \frac{2 u^4 \ln(u) \ln(s)}{s} + \frac{3}{2} u^3 \ln(u)^2 - \frac{9}{8} u^3 \ln(u) \right). \quad (3)$$

The s -channel has contributions from both particle types in the loop. The calculation for the fermion contribution is very similar to the u -channel case; as before (2) is the basic building block. Using this we find the cut contribution to be

$$-\frac{i \kappa^4 r_{\Gamma}}{(4\pi)^{2-\epsilon}} \frac{1}{16 \langle 24 \rangle [13]} \left(\frac{1}{\epsilon} \frac{u^2 (2u^2 + 3ut + 3t^2)}{t} \ln(s) - \frac{s^2 (2u^2 - t^2)}{t} \ln(s) \ln(t) - \frac{(t^3 + 2ut^2 + 2u^2t - u^3)}{2} \ln(s)^2 + \frac{u (274 ut - 53 u^2 + 60 t^2)}{60} \ln(s) \right). \quad (4)$$

The tree used for the cut with internal gravitons can be found using a previous result for the pure gravity tree [16,17] and SUSY Ward Identities. We can write it as

$$A(g^-, f^-, f^+, g^+) = \frac{\langle 13 \rangle^3}{\langle 12 \rangle^3} A(g^-, g^-, g^+, g^+) = \frac{i \kappa^2}{4} \frac{\langle 12 \rangle^3 \langle 13 \rangle^3}{\langle 23 \rangle^2 \langle 34 \rangle^2 \langle 14 \rangle^2} \frac{st}{u}. \quad (5)$$

Again, sewing two trees of this type together we obtain a cut result of

$$-\frac{i \kappa^4 r_{\Gamma}}{(4\pi)^{2-\epsilon}} \frac{1}{\langle 24 \rangle [13]} \left(\frac{u^4 \ln(s) \ln(u)}{8s} - \frac{1}{32} (4t^3 + 2s^3 + 11t^2s + 8ts^2) \ln(s)^2 - \frac{t^2 (2ts + 2t^2 + s^2) \ln(s) \ln(t)}{16s} - \frac{1}{960} u (7s^2 - 5ts + 48t^2) \ln(s) \right). \quad (6)$$

Summing (4) and (6), we see that the total s -channel cuts are

$$-\frac{i \kappa^4 r_{\Gamma}}{(4\pi)^{2-\epsilon}} \frac{u}{16 \langle 24 \rangle [13]} \left(\frac{1}{\epsilon} \frac{u^2 (2u^2 + 3ut + 3t^2)}{t} \ln(s) + \frac{1}{4} u (21s + 4t) \ln(s) + \frac{2 u^3 \ln(s) \ln(u)}{s} + \frac{(2s^4 + 2ts^3 - t^2s^2 + 2st^3 + 2t^4) \ln(s) \ln(t)}{st} + \frac{1}{2} (3s^2 + 10st + 6t^2) \ln(s)^2 \right). \quad (7)$$

We can find the t -channel contribution from this by making the exchange $s \leftrightarrow t$. If we combine the results from the three channels we can deduce the total expression. As expected, we have an IR contribution

$$-\frac{i \kappa^4 r_{\Gamma}}{2(4\pi)^{2-\epsilon}} \frac{1}{\epsilon} (u \ln(u) + s \ln(s) + t \ln(t)) \frac{1}{8} \frac{u^2 (st + 2t^2 + 2s^2)}{\langle 24 \rangle [13] st} \quad (8)$$

(see ref. [10] for details of the identification of IR contributions). Removing this leaves the UV component easily identifiable as

$$-\frac{59}{128} \frac{i\kappa^4 r_\Gamma}{(4\pi)^2} \frac{1}{\epsilon} \frac{u^3}{\langle 24 \rangle [13]}. \quad (9)$$

This expression must be cancelled by a four-point counterterm. This case has not been considered before, so we can use our result to deduce new information about the Dirac-Einstein counter-Lagrangian. The general on-shell counterterm which we must consider takes the form [7]

$$\frac{\alpha}{2} \frac{\kappa^4}{(4\pi)^2} \frac{e}{\epsilon} \left(\bar{\eta} \gamma_\mu \overleftrightarrow{\partial}_\nu \eta \right)^2. \quad (10)$$

where e is the determinant of the vierbein and α is a constant to be determined. (Note that other all other possibilities can be related to this by a combination of on-shell conditions, integration by parts and the Fierz theorem.) We can deduce that this will give a counterterm contribution of

$$\begin{aligned} & \frac{i\kappa^4}{(4\pi)^2} \frac{\alpha}{\epsilon} \left(\langle 2^+ | \gamma_\mu | 1^+ \rangle \langle 4^+ | \gamma^\mu | 3^+ \rangle (u-t) - \langle 4^+ | \gamma_\mu | 1^+ \rangle \langle 2^+ | \gamma^\mu | 3^+ \rangle (u-s) \right) \\ &= -6\alpha \frac{i\kappa^4}{(4\pi)^2} \frac{u^3}{\langle 24 \rangle [13]} \frac{1}{\epsilon}. \end{aligned} \quad (11)$$

Comparing this with (9) implies that

$$\alpha = -\frac{59}{768}. \quad (12)$$

The fact that this coefficient is non-zero is confirmation that the Dirac-Einstein system is non-renormalisable. Note that we have evaluated part of the counter-Lagrangian which could not be determined from the calculation in ref. [7].

Gravity Coupled to Photons

We could take the same route as above to find the infinities in photon amplitudes, but we can reduce the work by a judicious use of SUSY identities. (These have also proved useful in gauge theories one-loop computations [18].) Let us look here at the four point amplitude with no external gravitons. Although our aim is to find the infinities in a system containing photons and gravitons, let us begin by considering the $N = 2$ supersymmetric multiplet containing a photon a graviton and two gravitinos. The cut for this amplitude is the sum of the cuts with photons, gravitinos and gravitons in the loop, fig. B.

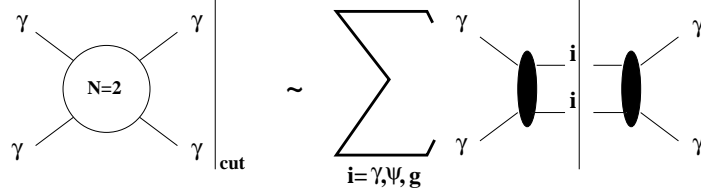


Figure B: Cut contributions required for the one loop four photon amplitude in an $N = 2$ SUGRA theory.

Notice that the tree amplitudes $A^{\text{tree}}(\gamma, \gamma, \gamma, \gamma)$ and $A^{\text{tree}}(g, g, g, g)$ do not include any contribution due to gravitinos (since any internal gravitinos must form a complete loop). Hence, the only gravitino contribution is contained in the second term in the sum in fig. B. We can deduce that the cuts for the amplitude in which we are interested, the one involving only gravitons and photons, are equal to the cuts of the $N = 2$ amplitude minus the cut containing gravitinos, fig. C.

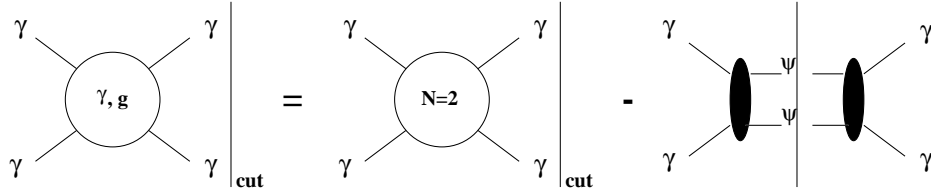


Figure C: Cut equation for the one loop four photon amplitude in Einstein-Maxwell theory.

Now, using supersymmetric relations between amplitudes we can make the following deductions: First, for choices of external helicities $(\gamma^- \gamma^- \gamma^+ \gamma^+)$ and $(\gamma^+ \gamma^+ \gamma^- \gamma^-)$ the gravitino contributions vanish and the cuts will be precisely equal to the cuts in the N amplitude. We know the $N = 2$ amplitude contains only IR divergences and hence there will be no UV infinite components in the photon-graviton amplitudes with those helicity choices.

Since we are only looking for UV divergences, the helicity configuration of interest to us here is $A(\gamma^- \gamma^- \gamma^+ \gamma^+)$. SUSY relations give us

$$A^{N=2}(\gamma^- \gamma^- \gamma^+ \gamma^+) = \frac{\langle 43 \rangle^2}{\langle 12 \rangle^2} A^{N=2}(g^- g^- g^+ g^+), \quad (13)$$

implying the relation in fig. D.

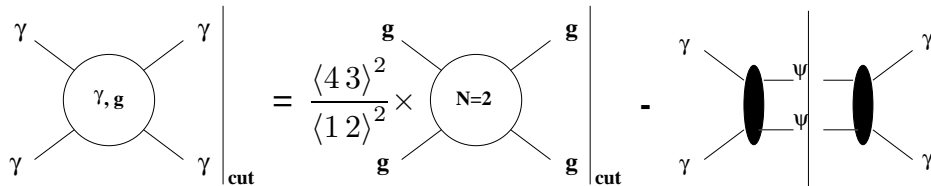


Figure D: Cut equation for the one loop four photon amplitude in Einstein-Maxwell theory (rewritten).

The one-loop amplitude $A^{N=2}(g^-g^-g^+g^+)$ has been calculated in ref. [19]. Using this we find that the cut contribution from the first term on the right hand side is

$$\begin{aligned}
F' \left(\frac{2}{\epsilon} \left(\frac{\ln(-u)}{st} + \frac{\ln(-t)}{su} + \frac{\ln(-s)}{tu} \right) \right. \\
+ \frac{2 \ln(-u) \ln(-s)}{su} + \frac{2 \ln(-t) \ln(-u)}{tu} + \frac{2 \ln(-t) \ln(-s)}{ts} \\
+ \frac{(3t^4 + 3t^3u - t^2u^2 + 3tu^3 + 3u^4) (\ln^2(-t/-u))}{s^6} \\
\left. + \frac{(t-u) (26t^2 + 46tu + 26u^2) \ln(-t/-u)}{s^5} \right)
\end{aligned} \tag{14}$$

where F' here is

$$\frac{i\kappa^4(4\pi)^\epsilon r_\Gamma}{16(4\pi)^2} \frac{\langle 43 \rangle^2}{\langle 12 \rangle^2} \left(\frac{st \langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \right)^2 = \frac{stu\kappa^2(4\pi)^\epsilon r_\Gamma}{4(4\pi)^2} A^{\text{tree}}(\gamma^-, \gamma^-, \gamma^+, \gamma^+). \tag{15}$$

It is clear that the divergences seen here are the complete IR contributions expected; there are no UV contributions from this ‘ $N = 2$ ’ part of the calculation. So, the UV divergence we are looking for will be found in the subtracted gravitino contribution.

In fact, we only need to carry out one calculation to find the divergences in the gravitino contribution: Note that if the helicities on one side of the cut are the same then at least one tree will vanish. Also, there is a symmetry between the two non-zero cut contributions – those in the t - and u -channels. So, if we calculate the t -channels divergences, we will have the complete result. Remarkably, using a combination of SUSY and previous results, we have been able to reduced the work required to a single calculation.

Let us look at the t -channel contribution. For this we will need the photon-gravitino tree. This can be found via another supersymmetric relation:

$$A^{N=2}(\gamma^-\psi^-\gamma^+\psi^+) = \frac{\langle 32 \rangle \langle 34 \rangle}{\langle 12 \rangle^2} A^{N=2}(g^-g^-g^+g^+). \tag{16}$$

Using this for a cut calculation gives us the result

$$\frac{i\kappa^4 r_\Gamma}{(4\pi)^{2-\epsilon}} \frac{s^4}{\langle 21 \rangle^2 [34]^2} \left(\frac{137}{960} \ln(t) + \frac{137}{960} \ln(u) - \ln^2(t/u) \right). \tag{17}$$

We must double this since there are two gravitinos in the SUSY multiplet; the contributions from both should be subtracted. So, we can deduce that the UV divergence is

$$\frac{137}{240} \frac{i\kappa^4}{(4\pi)^2} \frac{1}{\epsilon} \frac{s^4}{\langle 21 \rangle^2 [34]^2}. \tag{18}$$

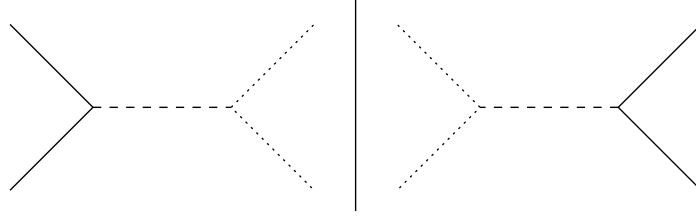


Figure E: Extra cut required when a second photon flavour is added to the theory.

We can compare this with the theoretical derivation of the counterterms (see ref. [5][†]); we confirm that these terms do, indeed, cancel this, as required.

We can also look at the effect of adding more (independent) $U(1)$ particles to the system. To add n more photons, we must consider n diagrams of the form fig. E, where the internal particle is a photon, but a different flavour to the external one. If we carry out such a calculation, and sum over all channels, we find that n extra photons produce an added UV infinite contribution of

$$\frac{n}{40} \frac{i\kappa^4}{(4\pi)^2} \frac{1}{\epsilon} \frac{s^4}{\langle 2\,1 \rangle^2 [3\,4]^2}. \quad (19)$$

Again, comparing this with the theoretical results [6], we find that the derived counterterms will remove this divergence.

In conclusion, we have shown how the Cutkosky rules can be used to find infinities in two quantum gravity theories. In the Dirac-Einstein case this enabled us to determine an unknown coefficient in the counter-Lagrangian. We were also able to confirm the previous derivation of Einstein-Maxwell counterterms. In the latter calculation we found that SUSY Ward identities could simplify the process significantly.

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[†] This actually differs from the result in ref. [5] by a factor of 4. This is simply due to a difference between their definition of κ and the definition implicit in our calculation.

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